Mesoscale structure of wrinkle patterns and defect-proliferated liquid crystalline phases

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Thin solids often develop elastic instabilities and subsequently complex, multiscale deformation patterns. Revealing the organizing principles of this spatial complexity has ramifications for our understanding of morphogenetic processes in plant leaves and animal epithelia and perhaps even the formation of human fingerprints. We elucidate a primary source of this morphological complexity—an incompatibility between an elastically favored “microstructure” of uniformly spaced wrinkles and a “macrostructure” imparted through the wrinkle director and dictated by confinement forces. Our theory is borne out of experiments and simulations of floating sheets subjected to radial stretching. By analyzing patterns of grossly radial wrinkles we find two sharply distinct morphologies: defect-free patterns with a fixed number of wrinkles and nonuniform spacing and patterns of uniformly spaced wrinkles separated by defect-rich buffer zones. We show how these morphological types reflect distinct minima of a Ginzburg–Landau functional—a coarse-grained version of the elastic energy, which penalizes nonuniform wrinkle spacing and amplitude, as well as deviations of the actual director from the axis imposed by confinement. Our results extend the effective description of wrinkle patterns as liquid crystals [H. Aharoni et al., Nat. Commun. 8, 15809 (2017)], and we highlight a fascinating analogy between the geometry–energy interplay that underlies the proliferation of defects in the mechanical equilibrium of confined sheets and in thermodynamic phases of superconductors and chiral liquid crystals.

Significance

Thin films readily buckle to relax compression, creating wrinkle patterns that may cover large portions of the solid. Predicting the arrangement of wrinkles in general settings is a major challenge that affects our understanding of stresses and deformations in textiles, biological tissues, and synthetic skins. We identify a mesoscale structure that arises from the incompatibility of uniformly spaced wrinkles and a spatially varying director, unraveling an organizing principle with analogs in liquid crystalline and superconducting states of matter.

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respectively. The stretching and bending moduli of the sheet are $Y = Et \gg \gamma_{\text{in}}, \gamma_{\text{out}}$ and $B = Et^3/12(1 - \Lambda^2)$, where $E, \Lambda$ are the Young’s modulus and Poisson’s ratio, respectively. The problem is governed by three dimensionless groups, to which we refer, respectively, as the “confinement,” tensional “bendability,” and a Bond-like parameter (i.e., the ratio between substrate stiffness and tensile stress):

$$\tau = \frac{\gamma_{\text{in}}}{\gamma_{\text{out}}}; \quad \epsilon^{-1} = \frac{\gamma_{\text{in}} R_{\text{in}}^2}{B}; \quad Bo = \frac{K_{\text{sub}} R_{\text{in}}^2}{\gamma_{\text{in}}}.$$  \hspace{1cm} \text{[1]}

We study thin, highly bendable sheets ($10^4 < \epsilon^{-1} < 10^6$). If $\tau$ exceeds a finite threshold ($\gtrsim 2$), hoop confinement emerges in an annulus, $R_{\text{in}} < r < L$, which expands upon increasing $\tau$, and compression is suppressed through azimuthal undulations, yielding radially oriented wrinkles. Macroscale features of the pattern are governed by $\tau$ (see below), whereas the wavelength scales as $\lambda(r) \sim \epsilon^{-1/4}$ ($10, 20, 22, 24$). Experiments exhibit a largely uniform wavelength when the substrate stiffness is strong ($Bo \gg 1$) (22), and a constant wrinkle number (i.e., $\lambda(r) \propto r$) for a weak substrate (20, 21, 25), suggesting that

$$\lambda(r)/R_{\text{in}} \approx \epsilon^{1/4}. \quad \left\{ \begin{array}{ll} 2\pi \cdot Bo^{-1/4} & \text{if } Bo \gg 1 \\ C(\tau) \cdot r / R_{\text{in}} & \text{if } Bo \ll 1 \end{array} \right.$$  \hspace{1cm} \text{[2]}

(where $C(\tau)$ is some smooth function).

Our experiments and simulations explore a broad range of $Bo$ values. The simulations employ a finite-element method, similarly to ref. 23, with a (locally hexagonal) disordered mesh to suppress spurious lattice effects on the pattern. The experiments use spin-coated, ultrathin polystyrene sheets (thickness

The images in Fig. 2 show two contrasting responses: a spatially constant wrinkle number (i.e., nonconstant wavelength, $\lambda \propto r$) and patterns with nearly constant wavelength ($\lambda \approx \lambda_0$, so that $m(r) \approx 2\pi r / \lambda_0$, as shown in Fig. 2 B and D with measurements of $m(r)$ in Fig. 2 F and H). The latter patterns are significantly more complex, as they require the proliferation of defects, i.e., points where wrinkles are created away from boundaries. These patterns also exhibit strong modulations of the wrinkle amplitude, such that defect-rich regions occur at amplitude-suppressed zones. Fig. 2F collects our observations into a phase diagram. The observed transition between defect-free and defect-rich states forms a curve $Bo_c(\tau)$ in the parameter plane ($\tau, Bo$), which we find to scale as

$$Bo_c(\tau) \sim (\tau - 2)^{-3.4 \pm 0.3}.$$  \hspace{1cm} \text{[3]}

Theory

The above findings motivate us to focus on the morphologically rich regime at large Bond number. This section and a subsequent one describe succinctly our theoretical approach, delegating many technical details to SI Appendix. Our starting point is “tension field theory” (TFT), which provides the leading-order Föppl–von Kármán (FvK) elastic energy at the singular limit of infinite bendability (24). Noting that the ratio between the compressive (hoop) and tensile (radial) components of the stress tensor must vanish as $\epsilon \to 0$, with in-plane force balance in the wrinkled zone ($R_{\text{in}} < r < L$), and matching to the purely tensile (unwrinkled) region yield

$$(R_{\text{in}} < r < L): \quad \sigma_{rr} \approx \gamma_{\text{in}} R_{\text{in}}/r; \quad L \approx R_{\text{in}} \tau/2.$$  \hspace{1cm} \text{[4]}

Suppression of hoop compression requires matching the contraction, $u_r / r$, due to the displacement $u_r(r)$ underlying [4], to the fraction of longitudinal modulus $\Phi^2$ “wasted” by wrinkles,

$$\Phi^2(\tau) \equiv \frac{1}{2\pi} \int_0^{2\pi} d\theta \left(\sqrt{1 + \left(\frac{\partial \Phi}{\partial \gamma}\right)^2} - 1\right) \approx \frac{1}{4\pi \tau^2} \int_0^{2\pi} d\theta \left(\frac{\Phi_r}{\gamma_{\text{in}}}\right)^2 \approx 2(\gamma_{\text{in}} / Y)(L/r) \log(L/r),$$  \hspace{1cm} \text{[5]}

where $\zeta(\theta, r)$ is the deflection from the plane (24). Notably, these macroscale features characterize the singular limit $\epsilon \to 0$, being indifferent to the wavelength $\lambda(r)$. The corresponding limit value of elastic energy (in comparison to a state with uniform strain $\gamma_{\text{out}} / Y$) does not depend on $\epsilon$ or $Bo$; namely,

$$U_{\text{down}}(\tau) \approx -\frac{\pi R_{\text{in}}^2}{2 \gamma_{\text{in}}} \cdot \frac{Y}{2} \log \frac{\tau}{2} \cdot \lambda - \frac{1}{2}.$$  \hspace{1cm} \text{[6]}

To remove this shape degeneracy, one must minimize an ($\epsilon$-dependent) contribution to the FvK energy that penalizes bending and substrate deformation, by solving the first FvK equation (assuming $\partial \Phi / \partial \gamma \gg (\partial \Phi / \partial \zeta)$),

$$\left(\mathcal{L}_0 + \mathcal{L}_1\right)\zeta(\theta, r) = 0$$  \hspace{1cm} \text{[7]}

where $\mathcal{L}_0 = B \frac{\partial^2}{\partial r^2} - \sigma_{\theta\theta} \frac{\partial^2}{\partial \theta^2} + K_{\text{sub}}$; $\mathcal{L}_1 = -\sigma_{\theta\theta} \frac{\partial^2}{\partial \theta^2}$. Here, $\sigma_{\theta\theta}$ is given by Eq. 4, and $\sigma_{\theta\theta}$ acts as a Lagrange multiplier that enforces the condition [5], analogous to the inextensibility constraint underlying one-dimensional (1D) elastica.
Specializing to a narrow annulus, \( r \in r_a \pm \ell_L \), where \( \ell_L \ll r_a \) and \( R_{in} < r < L \) (Fig. 3), and expecting the effect of \( \mathcal{L}_1 \) to vanish as \( Bo \to \infty \), we note that Eqs. 5 and 7 describe a collection of 1D, decoupled elastica rings of length \( 2\pi r \), subjected to confinement \( \Psi^2(r) \), Eq. 5, and substrate stiffness \( k_{sub} \). This motivates constructing a “monochromatic” ansatz from periodic elastica solutions, labeled by an integer \( m \):

\[
\zeta(r, \theta) \approx \Psi(r, \theta) g(\theta) \quad \text{with} \quad g(\theta) = \cos(m\theta); \quad \Psi(r) = 2r\Phi(r)/m, \quad [8]
\]

where \( \sigma \theta \theta(r) = -[B(m/r)^2 + K(m/r)^{-2}] \). [9]

The resulting energy of bending and substrate deformation is

\[
U_m \approx \tilde{C} + \frac{1}{2}B \int d\theta \int dr \Psi(r)^2 r^{-3} m^2 (m - \tilde{m}\{r\})^2 \quad [10]
\]

where \( \tilde{m}\{r\} \equiv 2\pi r / \lambda; \quad \lambda \equiv 2\pi \sqrt{B/k_{sub}} \). [11]

and \( \tilde{C} \) is a constant \( \approx \sqrt{BK_{sub}} \). Minimizing \( U_m \) over integers \( m \) yields \( m \approx m_a = 2\pi r_a \lambda \), recovering Eq. 2 for \( Bo \to \infty \).

The argument thus far is merely a reformulation of previous analyses of the wrinkle wavelength (1, 10), underscoring two difficulties. First, the ansatz [8] does not indicate how transitions occur between distinct values of \( m_a \) at adjacent narrow annuli. Second, for finite \( Bo \gg 1 \) the perturbation imposed by the operator \( \mathcal{L}_1 \) is resonant (i.e., \( L_1 \zeta \propto g(\theta) = \cos(m_a\theta) \) is a zero mode of \( \mathcal{L}_0 \)), akin to periodically driving a harmonic oscillator at its resonant frequency. Hence, Eq. 7 is impervious to regular expansion around the ansatz [8] unless \( \mathcal{L}_1 \zeta = 0 \Rightarrow \Psi''(r) = 0 \), which is incompatible with Eq. 5.

Motivated by multiscale perturbation theory of nonlinear dynamics problems, * we generalize the ansatz [8] to

\[
\zeta(r, \theta) \approx \text{Re}[\Psi(r, \theta)e^{im_a\theta}], \quad [12]
\]

where the complex amplitude \( \Psi(r, \theta) \) satisfies \( |\partial_\theta \Psi| \sim Bo^{-b} m_a |\Psi| \ll \Psi_0 |\Psi| \) for some \( b > 0 \). For the ansatz [12], the avoidance of resonant effects in a perturbative expansion of Eq. 7 around the limit \( Bo = \infty \) implies the equation

\[
|\sigma_{rr} \frac{\partial^2}{\partial r^2} + 4|\sigma_\theta \theta \frac{\partial^2}{\partial r \partial \theta} |\Psi(r, \theta) = 0, \quad [13]
\]

such that the only nontrivial value of the exponent \( b \), for which the two terms in Eq. 13 are in balance for \( Bo \gg 1 \), is \( b = 1/2 \) (where we used Eqs. 9 and 11). In contrast to the ansatz [8], Eq. 13 admits azimuthally oscillatory solutions to the wrinkle amplitude. Crucially, this facilitates a mechanism for transitioning between distinct integer values of \( m_a \) at adjacent annuli: Defects can nucleate within amplitude-suppressed zones (\( \Psi(r, \theta) \approx 0 \)) at negligible energy cost.

Within a narrow, defect-free annulus around \( r = r_a \), an azimuthally oscillatory solution to Eq. 13 satisfies \( \Psi \propto \exp[-2\pi(r - r_a)\ell_\parallel^{-1} \sqrt{\sigma_\theta \theta/r_{in}} \cos(\pi r / \ell_\parallel)] \). A crude estimate of \( \ell_\parallel \), over which the amplitude varies azimuthally, may be obtained by requiring \( \partial_\theta \Psi / \partial r \sim \Psi / \ell_\parallel \) to ensure compatibility with Eq. 5, and recalling Eqs. 9 and 11.

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* Such problems often feature a slow temporal modulation of the amplitude of a periodic signal (e.g., \( \Delta \phi \cos(\omega t) \rightarrow \text{Re}[A(t)e^{i(\omega t + \Delta \omega t^2)}] \), such that \( |\Delta \omega | \ll \frac{\Delta \phi^{(1)}}{2\pi} |\omega| \).

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Fig. 3. (A) Schematic of a narrow, defect-free annular zone, where Eq. 12 holds. (B) Schematic of the deflection \( \zeta(r, \theta) \) at a given \( r \), exhibiting rapid undulations of wavelength \( \approx \lambda \), modulated by the slowly varying (complex) amplitude \( \Psi(r, \theta) \).
\[ \ell_0 \approx \ell_{BC} \cdot r_a / \lambda \quad \text{where} \quad \ell_{BC} \equiv \sqrt{B / \sigma_{\tau}(r_a)}. \]  

Here, \( \ell_{BC} \) is a (local) “bendo-capillary” length (26).

### Coarse-Grained Energy and Amplitude Modulations

Let us now elaborate on the energetic hierarchy imparted by the ansatz (12) and derive a generic, quantitative version of Eq. 14. Exploiting the separation of scales between the wrinkly undulations and their slowly varying amplitude, we derive in SI Appendix a Ginzburg–Landau–like energy functional for the amplitude \( \Psi(r, \theta) \), by averaging out the FvK energy over the small scale \( \lambda \). This coarse-graining calculation recovers the energy \( U_{\text{nonlin}} \) (Eq. 10) and yields two other terms,

\[ U_{\text{nonlin}} \approx \frac{Y}{2} \int d\theta \int r d\theta \left[ \frac{m^2}{4\epsilon^2} - \Phi^2(r) \right]^2 \right] + \frac{m^2}{4\epsilon^2} \left( \frac{\partial \Phi}{\partial \theta} \right)^2, \]

\[ U_{\Psi} \approx \frac{\lambda}{2} \int d\theta \int r d\theta \left[ \sigma_{\tau}(\theta) \right] \left( \frac{\partial \Phi}{\partial \theta} \right)^2 \]  

where \( \Phi, \sigma_{\tau}, \sigma_{\theta} \) are given by Eqs. 4, 5, and 9. The sum \( U_{\text{nonlin}} + U_{\tau} + U_{\psi} \) describes the deviation of the energy of an actual wrinkle pattern from the TFT limit value (Eq. 6), reflecting a hierarchy of energetic costs. \( U_{\text{nonlin}} \) (Eq. 15) includes quartic terms in \( \Psi \), which must be considered since they are penalized by a large stretching modulus, \( Y \), reflecting strain in excess of the residual level already accounted for in the limit value (Eq. 6). The term \( U_{\tau} \) (given by Eq. 10 above) reflects a balance between bending and substrate (i.e., liquid gravity) energies, yielding the favored wrinkle wavelength, \( \lambda \). The last term, \( U_{\psi} \), reflects the energy cost for amplitude modulations and thereby the emergence of defects to enable proximity of the wavelength \( \lambda(r) \) to \( \lambda \) throughout the pattern. Its typical value is \( \mathcal{O}(\lambda^{-1}) \) relative to the energy incurred by \( U_{\text{nonlin}} \), and its corresponding Euler–Lagrange equation is precisely Eq. 13.

For a small, defect-free zone of azimuthal and radial extents \( \ell_x, \ell_\perp \sim r_a \), the second term in Eq. 15 indicates that strain is induced by any deviation of wrinkles from the tension-carrying lines (i.e., radial lines, for which \( \partial \Phi / \partial \theta = 0 \)). Hence, the radial orientation of wrinkles persists in defect-free zones, locally suppressing the smectic order imparted by uniformly spaced wrinkles (for which \( \partial \Phi / \partial \theta = \pm \tilde{m}(r) - m_n \)). In contrast, the first term in Eq. 15 does not vanish for any azimuthally oscillating amplitude, but its cost can be made negligible (energy density \( \sigma_{\tau}(\theta) \)). The second term in \( U_{\text{nonlin}} \) includes quartic \( \Phi \) terms in \( \Phi \), reflecting a balance of bend-induced \( \parallel \) and small \( \perp \) director distortions. Hence, the radial deviation from the mean orientation of the smectic order imparted by \( \Phi \) yields the characteristic distance \( \ell_\perp \) between the defect-riddle pattern and \( \Phi \) in experiments (10) and the vertical deflection \( \zeta(\theta) \) in simulations. A pronounced wrinkle wavelength is clearly present, with large amplitude modulations. Fig. 4 B and D shows the measured lengthscale of these modulations at several radii (normalized by the measured \( \lambda \)), and Fig. 4E compares measured and predicted values of \( \ell_\perp / \lambda \) for several values of \( Bo \) and \( \tau \). No fitting parameters are used. The experiments exhibit good agreement with the theoretical predictions, whereas the simulation results are systematically lower than expected and approach the predicted values only at large \( Bo \) and small \( \tau \), where the extent of the wrinkled zone is relatively small (Eq. 4). We attribute this discrepancy to the numerical difficulty in reaching the most low-lying energy states. In particular, at large \( \tau \) we find many metastable states where defects are scattered throughout the wrinkled region, thereby increasing the characteristic distance \( \ell_\perp \) between the defect-rich zones.

### General Form of Theoretical Results

The arguments underlying Eq. 17 apply to a broad class of confinement problems, for which a wavelength \( \lambda(x) \) is favored in the vicinity of a point \( x \) on the sheet by competition of bending resistance and substrate-induced stiffness (akin to Eqs. 2 and 11). For given boundary loads and substrate shape, a TFT solution may be found analytically or numerically (27), yielding the macroscopic fields—a director \( \tilde{n}(x) \), along which wrinkles “waste” a fraction of arclength \( \Phi_{\text{macro}}(x) \), and a tensile stress \( \sigma_{\tau}(x) \) (akin to Eqs. 5 and 4, respectively). Here, “aux” refers to an auxiliary state that describes the singular, infinite bendability limit, of a hypothetical body with zero bending modulus, and \( \parallel, \perp \) denote the (curvilinear) planar axes, along and normal to \( \eta_{\text{aux}}(x) \). Our analysis predicts that if \( \eta_{\text{aux}}(x) \) is bent (\( \nabla \times \eta_{\text{aux}} \neq 0 \)), then a defect-riddled pattern consists of defect-free domains where \( \lambda(x) \approx \tilde{\lambda}(x) \), whose longitudinal scale, \( \ell_\parallel \), is given by Eq. 17, with

\[ \ell_{BC} = \sqrt{\frac{B}{\sigma_{\tau}(x)}}, \quad \ell_{\text{bend}}^{-1} = \left[ \nabla \times \eta_{\text{aux}}(x) \right] + \left[ \nabla \Phi_{\text{aux}}(x) \right]. \]  

The applicability of our result to another axial confinement problem is demonstrated in Fig. 1 C, which shows wrinkles near the edge of a circular sheet of radius \( W \) attached to a spherical Wrinkler foundation [a ball of springs of stiffness \( \propto \kappa_{\text{e}} \) and rest length \( R \gg W \) (17, 18)]. For that problem, the tensional bendability \( \epsilon_r \) and Bond number \( Bo \) are defined similarly to Eq. 1, with \( \gamma_{\text{macro}} \) replaced by the tensile load \( -\kappa_{\text{e}} \) exerted at the perimiter, and the parameter analogous to \( \tau \) is \( \alpha = \frac{\epsilon_r^2}{\kappa_{\text{e}}} \), which controls the strength of azimuthal confinement. A TFT solution (section IV A–C in ref. 17) yields expressions for \( \sigma_{\tau}(r) \) and \( \Phi(r) \) analogous to Eqs. 4 and 5, which we substitute in Eq. 18 to compute \( \ell_\parallel \) at \( r = W \) (scale bar in Fig. 1 C).

### Comparison with Experiments and Simulations

Fig. 4 A and C shows the azimuthal profile of a sheet with \( Bo > Bo_0 \) at a single radius, which we quantify using the image intensity \( I(\theta) \) in experiments (10) and the vertical deflection \( \zeta(\theta) \) in simulations. A pronounced wrinkle wavelength is clearly present, with large amplitude modulations. Fig. 4 B and D shows the measured lengthscale of these modulations at several radii (normalized by the measured \( \lambda \)), and Fig. 4E compares measured and predicted values of \( \ell_\perp / \lambda \) for several values of \( Bo \) and \( \tau \). No fitting parameters are used. The experiments exhibit good agreement with the theory, whereas the simulation results are systematically lower than expected and approach the predicted values only at large \( Bo \) and small \( \tau \), where the extent of the wrinkled zone is relatively small (Eq. 4). We attribute this discrepancy to the numerical difficulty in reaching the most low-lying energy states. In particular, at large \( \tau \) we find many metastable states where defects are scattered throughout the wrinkled region, thereby decreasing the characteristic distance \( \ell_\perp \) between the defect-rich zones.

### Geometric Conflicts and Defect-Proliferated States

To place our study in a broader context, we consider a thermodynamic ensemble of elongated molecules (“nematogen”). Upon cooling or increasing density, this prototypical system transforms from an isotropic liquid to a “nematic” phase, where the molecular axes are parallel on average. The corresponding order parameter, reflecting broken rotational symmetry, is a director field with a uniform ground state (\( \tilde{n}(x) = \tilde{n}_0 \)). Cooling further leads to a “smectic-A” phase, where molecules form uniformly

\[ \ell_0 \approx \ell_{BC} \cdot r_a / \lambda \quad \text{where} \quad \ell_{BC} \equiv \sqrt{B / \sigma_{\tau}(r_a)}. \]  

In addition to replacing the scaling relation in Eq. 14 by a number (4\( \pi \)), the length \( r_a \) is replaced by yet another local length \( \ell_{\text{bend}} \), which derives from the (planar) curvature of the axis \( \theta \) along which wrinkles suppress an imposed compression.
The broken translational symmetry underlies a complex order near localized grain boundaries (i.e., grain boundary (TGB) phase (15)—an inhomogeneous ground state of a twisted director (\(\theta\)). Here, the nematic is replaced by a “cholesteric” phase, penalizing \(\omega = \ell_r \theta\) measured at \(r/R_{in}\) from the experiment in Fig. 2B with \(B_0 = 0.25\) and \(\tau = 10\). The signal oscillates with a wavelength \(\lambda\), and the amplitude oscillates over a longer scale, \(\ell_r\). (B) Radial dependence of \(\ell_r/\lambda\) measured in the same experiment (solid circles). (C) Vertical displacement \(z(\theta)\) at \(r/R_{in} = 2.4\) in a simulation with \(B_0 = 27\) and \(\tau = 6\). (D) Radial dependence of \(\ell_r/\lambda\) extracted from the same simulation (open circles). In B and D the solid curves show the prediction of Eq. 17. (E) In the main plot, measured \(\ell_r/\lambda\) versus the predicted value, given by Eq. 17. For sheets subjected to larger \(\tau\) (and hence having longer wrinkles), we measure \(\ell_r/\lambda\) at multiple radii. The experiments are in good agreement with the theory, whereas simulations agree at large \(B_0\) and small \(\tau\) (main text).

Note that although the experimental system has \(B_0 < 1\), the local parameter \(B_0 = K_{aux} \ell_r^2 / \sigma_r (\ell_r)\), whose inverse is the expansion parameter in our theory (Appendix), is substantially larger than 1 in most of the wrinkled zone. (Inset) The measured wrinkled number agrees with the gravity-dominated value, \(m(r) = (r \mu / B)^{1/4}\) (solid line). Data in Inset are sampled at the same radii as in the main plot.

The crucial role of the director in 3D liquid-crystal phases and 2D wrinkled sheets motivates us to further develop the analogy between these systems (28). When the energetics favor a constant director, a defect-free layered structure may emerge: the smectic-A phase for nonchiral nematogen or a parallel array of uniform wrinkles for a uniaxially confined sheet. In the geometrically conflicted case of a nonconstant director (twisted for chiral nematogen and bent for an azimuthally confined sheet), one finds an inhomogeneous, defect-proliferated structure that retains parallel layers in separated domains: the TGB phase or a defect-rich wrinkled film.

This analogy is bolstered by contrasting our coarse-grained energy, \(U_m + U_\phi + U_{\text{nonlin}}\) (Eqs. 10, 15, and 16), and that of chiral nematogen (equation 2.11 of ref. 15). In our 2D wrinkled sheet, the energy \(U_m\) favors smectic order, whereas the confining forces favor a bent director, \(\hat{n}_\text{aux} = \hat{n}\), through the energy \(U_{\text{nonlin}}\). If the energetic penalty of \(U_m\) is small, the director \(\hat{n}_\text{aux}\) is imposed forcefully, precluding smectic order (i.e., \(\lambda (r) \approx \lambda\)) in analogy with the cholesteric phase (\(B_0 < B_0\), in Fig. 2). In contrast, if \(B_0 > B_0\), proliferation of defect-rich, amplitude-suppressed zones enables a partial recovery of smectic order (i.e., \(\lambda (r) \approx \lambda\)) in defect-free domains, in analogy with the TGB phase. The parameter \(B_0\) is thus akin to a Frank modulus, which penalizes deviations of the director’s twist from the value \(\hat{n}_r\), imparted by a nematogen chirality. Furthermore, the size (“coherence length”) of smectic domains in the TGB phase derives from the minimal energy associated with varying the order parameter (\(|\Psi|: 0 \rightarrow A_0\)) between defect-rich planes and defect-free domains. This is again similar to our system, where the azimuthal extent \(\ell_r\) of defect-free zones (Eq. 17) derives from the energy (\(U_\phi + U_{\text{nonlin}}\)) required to generate regions where the wrinkle amplitude \(|\Psi|\) is suppressed.

### Discussion

Wrinkle patterns—highly curved periodic undulations that waste an excess length—are common in strongly confined thin solids that are forced to reside close to a smooth substrate. For such problems, tension field theory or its recent extension (18) predicts a macroscale thickness-independent director field \(\hat{n}_\text{aux}(x)\), a macroscale wavelength \(\lambda(x)\), and a corresponding stress field. The simplest pattern—a smectic-like array of uniformly spaced, parallel wrinkles—emerges when both \(\hat{n}_\text{aux}(x)\) and \(\lambda(x)\) are constants. However, if either field is spatially varying, it may be impossible for the confined body to satisfy both fields everywhere, and the ensuing negotiation gives rise to a host of mesoscale morphologies.

Our work addresses conformations characterized by a uniform \(\lambda\) and a purely bent director, (i.e., \(\nabla \cdot \hat{n}_\text{aux} = 0, \nabla \times \hat{n}_\text{aux} \neq 0\)). Studying the Lamé setup as a prototypical model for such problems, we showed that the pattern may either consist of a fixed number of wrinkles, absent of smectic order, or be
characterized by amplitude modulations over a mesoscale $\ell_j$ (Eq. 17) that enable proliferation of defect-rich zones and thereby partial recovery of smectic order (lower and upper parts of Fig. 2f, respectively). A different type of confinement with uniform $A$ and nonuniform but unbent director ($\nabla \times \hat{n}_{aux} \approx 0$) yields a qualitatively different mesoscale structure, an example of which was realized by forcing a patch of a spherical shell to reside close to a plane (28, 29) (Fig. 1B). Here, the unbent director is piecewise constant, containing splay ($\nabla \cdot \hat{n}_{aux} \neq 0$) only at “domain walls” that separate defect-free, smectically ordered domains of uniformly spaced wrinkles (28). Two other notable confinement types may occur even under uniaxial compression (e.g., $\hat{n}_{aux} \approx \hat{n}$), if the locally favorable wavelength is spatially varying such that $\nabla \times \hat{n}_{aux} \neq 0$ or the wrinkle amplitude itself is forced to vary spatially such that $\nabla \Psi \times \hat{n}_{aux} \neq 0$. A recent experimental study that addressed the former case employed a sheet with a nonuniform thickness (where patterns of defects that resemble Fig. 2 C and D were observed) (30), whereas the latter type, which underlies “wrinkle cascades” (Fig. 1A), was realized through amplitude-suppressing boundary conditions (16, 31).

While our theoretical analysis pertains to confinement problems with bent director fields ($\nabla \times \hat{n}_{aux} \neq 0$), we anticipate that the coarse-graining approach initiated in ref. 28 and further developed here may provide a unified framework to analyze mesoscale structures in a broad class of confinement problems (4). At its core there is an energy functional of a slowly varying complex function, whose magnitude describes the wrinkle amplitude and whose phase describes deviations from an asymptotic (thickness-independent) director field $\hat{n}_{aux}$, imposed by the confining forces. Such a Ginzburg–Landau-like functional may be obtained by expanding the full elastic energy around a suitable TFT limit and coarse graining over the wrinkle microscale. Pursuing this approach further may reveal additional analogies with liquid crystals and superconductors and elucidate the remarkable complexity of wrinkle patterns.

Materials and Methods

Experiments. Polymer films were made by spin-coating solutions of polystyrene ($M_N = 99,000, M_M = 105,500$; Polymer Source) in toluene (99.9%; Fisher Scientific) onto glass substrates following ref. 20. A white-light interferometer (Filmetrics F3) was used to measure film thickness, which was uniform over each film to within 4%. Methods for determining $\hat{n}_{aux}$ and descriptions of our image analysis routines are provided in SI Appendix.

Simulations. Finite-element simulations were performed in ABAQUS, as detailed in SI Appendix.

Data Availability. The data that support the findings of this study are available in Datasets 51–53.

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